

ANISOTROPIC ISOPARAMETRIC HYPERSURFACES IN EUCLIDEAN SPACES

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ABSTRACT. In this note, we give a classification of complete anisotropic isoparametric hypersurfaces, *i.e.*, hypersurfaces with constant anisotropic principal curvatures, in Euclidean spaces, which is in analogue with the classical case for isoparametric hypersurfaces in Euclidean spaces. On the other hand, by an example of local anisotropic isoparametric surface constructed by B. Palmer, we find that anisotropic isoparametric hypersurfaces have both local and global aspects as in the theory of proper Dupin hypersurfaces.

1. INTRODUCTION

Let $F : \mathbb{S}^n \rightarrow \mathbb{R}^+$ be a smooth positive function defined on the unit sphere satisfying the following convexity condition:

$$(1.1) \quad A_F := (D^2F + FI)_u > 0,$$

for any $u \in \mathbb{S}^n$, where D^2F denotes the Hessian of F on \mathbb{S}^n , I denotes the identity on $T_u\mathbb{S}^n$ and > 0 means the matrix is positive definite. Let $x : M \rightarrow \mathbb{R}^{n+1}$ be an immersed oriented hypersurface without boundary and $\nu : M \rightarrow \mathbb{S}^n$ denote its Gauss map. Then *anisotropic surface energy* (of x) is a parametric elliptic functional \mathcal{F} defined as follows:

$$\mathcal{F}(x) = \int_M F(\nu) dA.$$

Note that if $F \equiv 1$, then $\mathcal{F}(x)$ is just the area of x .

The critical points of \mathcal{F} for all compactly supported volume-preserving variations are characterized by the property that the *anisotropic mean curvature* H_F is constant, where H_F is given by

$$nH_F = nHF - \operatorname{div}_M DF = -\operatorname{tr}_M d(\phi \circ \nu).$$

Here ϕ is defined in (1.3) below and

$$(1.2) \quad S_F := -d(\phi \circ \nu) = -A_F \circ d\nu = A_F \circ T \circ dx$$

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is called the F -Weingarten operator¹ with respect to the induced metric of x , where $T = -d\nu$ is the Weingarten (shape) operator of M . In general S_F is not self-dual, but it still has real eigenvalues $\lambda_1, \dots, \lambda_n$, which are called *anisotropic principal curvatures*. If $\lambda_1 = \lambda_2 = \dots = \lambda_n$ holds everywhere on M , M is called *totally anisotropic umbilical* for F or for \mathcal{F} . Similarly, M is called *anisotropic isoparametric* for F or for \mathcal{F} , if its anisotropic principal curvatures are constant.

A fundamental result relating to the anisotropic surface energy is the Wulff's theorem, which states that among all closed hypersurfaces enclosing the same volume, there exists an absolute minimizer W_F of \mathcal{F} (cf. [1], [14]). Here W_F is the so-called "Wulff shape" that can be defined as follows. Consider the map

$$(1.3) \quad \begin{aligned} \phi : \mathbb{S}^n &\rightarrow \mathbb{R}^{n+1} \\ u &\mapsto DF_u + F(u)u, \end{aligned}$$

where DF is the gradient of F on \mathbb{S}^n . Then $W_F := \phi(\mathbb{S}^n)$ is called the *Wulff shape* of F or \mathcal{F} (cf. [6], [4], [16], etc.). Under the convexity condition of F , W_F is a smooth convex hypersurface. When $F \equiv 1$, the Wulff shape is just the unit sphere \mathbb{S}^n . An equivalent definition of the Wulff shape can be given in terms of the dual norm F^* of F , where $F^* : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ is defined by ([7])

$$F^*(y) = \sup \left\{ \frac{\langle y, z \rangle}{F(z)} \mid z \in \mathbb{S}^n \right\}.$$

Thus the Wulff shape W_F is just the unit sphere under this norm F^* , i.e.,

$$W_F = \{y \in \mathbb{R}^{n+1} \mid F^*(y) = 1\}.$$

The Wulff shape plays the same role as the unit sphere for the area functional. For example, up to translations and homotheties, W_F is the only closed stable (for the functional \mathcal{F}) hypersurface immersed in \mathbb{R}^{n+1} with constant anisotropic mean curvature ([12],[8]), and also the only closed embedded hypersurface with constant anisotropic mean curvature ([7],[10]). Moreover, in the case of dimension 2, W_F is the only topological 2-sphere immersed in \mathbb{R}^3 with constant anisotropic mean curvature ([9]).

As to the classification of totally anisotropic umbilical hypersurfaces, we have the following

Proposition 1.1. (cf. [7], [12]) *Any totally anisotropic umbilical hypersurface immersed in \mathbb{R}^{n+1} ($n \geq 2$), or $\lambda_1 \equiv \text{constant} \neq 0$ ($n = 1$), is an open part of a hyperplane or the Wulff shape, up to translations and homotheties.*

Comparing with the classical hypersurface theory, it is natural to ask the classification problem of anisotropic isoparametric hypersurfaces in Euclidean spaces. In this

¹For simplicity, sometimes we write the operators without dx or identify them with corresponding matrix representations.

note, we will give such a classification which was conjectured to be parallel with the classical case of isoparametric hypersurfaces in Euclidean spaces.

For each totally geodesic k -dimensional sphere $\mathbb{S}^k \subset \mathbb{S}^n$, we denote by $W_F^k := \phi(\mathbb{S}^k) \subset W_F$ being its image under the map ϕ defined in (1.3). It is easily seen that W_F^k is a k -dimensional submanifold of the Wulff shape W_F and depends on the choice of the inclusion $\mathbb{S}^k \subset \mathbb{S}^n$. Note that in general W_F^k may not equal to either the Wulff shape $\overline{W_F^k}$ of $F|_{\mathbb{S}^k}$ in $\mathbb{R}^{k+1} \subset \mathbb{R}^{n+1}$, or the intersection of W_F with a $(k+1)$ -dimensional hyperplane. However, there is a family of natural embeddings of $W_F^k \times \mathbb{R}^{n-k}$ in \mathbb{R}^{n+1} as follows. For any $t \neq 0$, define

$$(1.4) \quad \begin{aligned} \varphi_t : W_F^k \times \mathbb{R}^{n-k} &\rightarrow \mathbb{R}^{n+1} \\ (\phi(u), v) &\mapsto v - t\phi(u), \end{aligned}$$

where $u \in \mathbb{S}^k \subset \mathbb{R}^{k+1}$ and $\mathbb{R}^{n-k} \subset \mathbb{R}^{n+1}$ is the orthogonal subspace of the given \mathbb{R}^{k+1} . It is not hard to check that φ_t is an immersion and has two distinct constant anisotropic principal curvatures $1/t, 0$ with multiplicities $k, n-k$ respectively. In fact, φ_t is an embedding and its image coincides with that of the natural inclusion $\overline{W_F^k} \times \mathbb{R}^{n-k} \subset \mathbb{R}^{n+1}$ up to translations and homotheties, since the image of the projection of W_F^k to \mathbb{R}^{k+1} is just the Wulff shape $\overline{W_F^k}$. Then our main result can be stated as follows.

Theorem 1.1. *A complete hypersurface in the Euclidean space \mathbb{R}^{n+1} has constant anisotropic principal curvatures if and only if up to translations and homotheties, it is*

- (i) $\mathbb{R}^n \subset \mathbb{R}^{n+1}$, or
- (ii) $W_F \subset \mathbb{R}^{n+1}$, or
- (iii) $W_F^k \times \mathbb{R}^{n-k} \xrightarrow{\varphi_t} \mathbb{R}^{n+1}$ for some $0 < k < n$, $t \neq 0$, and $\mathbb{S}^k \subset \mathbb{S}^n$.

Note that when $F \equiv 1$, this theorem reduces to the well-known classification of isoparametric hypersurfaces in Euclidean spaces. For the history and recent progresses of isoparametric hypersurfaces in real space forms, we would like to refer to the excellent surveys given by Thorbergsson [15] and Cecil [2]. From them one can also find introductions of many important generalizations of isoparametric hypersurfaces such as (proper) Dupin hypersurfaces. In contrary with isoparametric hypersurfaces, proper Dupin hypersurfaces have both local and global aspects, such as the number g of distinct principal curvatures could have different sets of values for local and compact proper Dupin hypersurfaces, which happens to appear also in our situation. In fact, in a letter to us B. Palmer [13] showed that there is a rotationally symmetric anisotropic surface energy functional \mathcal{F} (or corresponding F) such that a part U of the helicoid Σ in \mathbb{R}^3 has constant anisotropic principal curvatures ± 1 . Then we can extend the corresponding F to a positive function \tilde{F} on \mathbb{S}^3 such that locally it equals F along the normal geodesics starting from the Gauss image \mathcal{U} ($\subset \mathbb{S}^2 \subset \mathbb{S}^3$) of U , i.e.,

$$\tilde{F}(\cos tu + \sin te_4) := F(u), \quad \text{for any } u \in \mathcal{U}, t \in (-\varepsilon, \varepsilon),$$

where e_4 is the fourth coordinate vector of $\mathbb{R}^4 = \mathbb{R}^3 \oplus \mathbb{R}$. It is easily verified that the function \tilde{F} satisfies the convexity condition (1.1) on a small neighborhood of \mathcal{U} in \mathbb{S}^3 (and thus could be extended to the whole \mathbb{S}^3 satisfying (1.1)). For this function \tilde{F} , one can see that (a part of) the canonical embedding $U \times \mathbb{R} \hookrightarrow \mathbb{R}^4$ has three constant anisotropic principal curvatures 1, -1 and 0. Hence, one can obtain local anisotropic isoparametric hypersurfaces in \mathbb{R}^{n+1} with more than 2 distinct anisotropic principal curvatures, which differs with the complete case as classified in Theorem 1.1.

2. ANISOTROPIC PARALLEL TRANSLATION

In this section, we will study the geometry of “anisotropic parallel translation” of hypersurfaces in \mathbb{R}^{n+1} and apply it to establish some preliminary results. Throughout of this paper, we will use the same notations as those in Section 1.

Let $x_t : M \rightarrow \mathbb{R}^{n+1}$ be a family of small perturbations of x ($x_0 = x$) defined by

$$(2.1) \quad x_t = x + t\phi \circ \nu.$$

Comparing with standard parallel translation, we call such perturbation x_t an *anisotropic parallel translation*. Taking differential of (2.1), we get

$$(2.2) \quad dx_t = (I - tS_F) \circ dx.$$

Thus, for any t small enough, x_t remains an immersion and ν (up to parallel translations) is still the unit normal vector field of x_t . Then by (1.2) the F -Weingarten operator for x_t , say S_F^t , can be expressed as

$$(2.3) \quad S_F^t = S_F \circ dx = S_F \circ (I - tS_F)^{-1} \circ dx_t,$$

where $(I - tS_F)^{-1}$ is the inverse of the operator $(I - tS_F)$. If $S_F X = \lambda X$, this becomes $S_F^t X = \frac{\lambda}{1-t\lambda} X$. Therefore, the anisotropic principal curvatures of x_t , say $\lambda_1(t), \dots, \lambda_n(t)$, can be expressed in terms of those of x as follows:

$$(2.4) \quad \lambda_i(t) = \frac{\lambda_i}{1-t\lambda_i}, \quad i = 1, \dots, n.$$

By standard method as Cartan and Nomizu did (cf. for example, [11],[3]), from (2.4) we can similarly obtain the following result.

Theorem 2.1. *Let $x_t : M \rightarrow \mathbb{R}^{n+1}$, $-\varepsilon < t < \varepsilon$, be a family of anisotropic parallel hypersurfaces in \mathbb{R}^{n+1} . Then $x(M)$ has constant anisotropic principal curvatures if and only if each $x_t(M)$ has constant anisotropic mean curvature.*

Proof. For the sake of completeness, we give a simple proof as follows.

Let $H_F(t)$ be the anisotropic mean curvature of $x_t(M)$. Then by (2.4), we have

$$\begin{aligned}
 nH_F(t) &= \text{tr}_{x_t(M)} S_F^t = \sum_{i=1}^n \lambda_i(t) = \sum_{i=1}^n \frac{\lambda_i}{1 - t\lambda_i} \\
 &= \sum_{i=1}^n (-\log(1 - t\lambda_i))' = - \left(\log \left(\prod_{i=1}^n (1 - t\lambda_i) \right) \right)' \\
 (2.5) \quad &= - \left(\log \left(\sum_{k=0}^n (-1)^k M_k t^k \right) \right)',
 \end{aligned}$$

where $'$ denotes taking derivative with respect to t , and $M_0 \equiv 1, M_1, \dots, M_n$ are the elementary symmetric polynomials of $\lambda_1, \dots, \lambda_n$. Then we can get our conclusions from (2.5) directly, since it implies that $H_F(t) \equiv \text{const}$ on $x_t(M)$ for each $-\varepsilon < t < \varepsilon$ is equivalent to $M_k \equiv \text{const}$ on $x(M)$ for $k = 1, \dots, n$ and hence is equivalent to $\lambda_i \equiv \text{const}$ on $x(M)$ for $i = 1, \dots, n$. \square

Now suppose $x(M)$ has distinct constant anisotropic principal curvatures $\lambda_1, \dots, \lambda_g$ with multiplicities m_1, \dots, m_g , i.e., it is anisotropic isoparametric. Let

$$D_i(p) := \{X \in \mathcal{T}_p M \mid S_F X = \lambda_i X\}, \quad p \in M.$$

Then we obtain distributions D_1, \dots, D_g of dimensions m_1, \dots, m_g on M . Similarly as in standard isoparametric hypersurface theory, we establish the following.

Proposition 2.1. *Each distribution D_i is integrable and when $\lambda_i \neq 0$, the maximal integral manifold $L_i(p)$ of D_i through $p \in M$ is, up to translations and homotheties, just a $W_F^{m_i}$ for some $\mathbb{S}^{m_i} \subset \mathbb{S}^n$, or just an open part of it, if M is not complete.*

Proof. Recall that for a hypersurface M in a real space form the Codazzi equation takes the form:

$$(\nabla_X T)Y = (\nabla_Y T)X, \quad \text{i.e.,}$$

$$(2.6) \quad \nabla_X(TY) - T(\nabla_X Y) = \nabla_Y(TX) - T(\nabla_Y X),$$

for X, Y tangent to M , where T is the shape operator of M and ∇ is the Levi-Civita connection. Due to the fact that D_i is trivially integrable if $m_i = 1$, we assume $m_i \geq 2$ in the following. Let X, Y be linearly independent local vector fields in D_i . Then (2.6) becomes

$$\nabla_X \lambda_i(A_F^{-1}Y) - T(\nabla_X Y) = \nabla_Y \lambda_i(A_F^{-1}X) - T(\nabla_Y X),$$

which becomes

$$(2.7) \quad \lambda_i(\nabla_X(A_F^{-1}Y) - \nabla_Y(A_F^{-1}X)) = T[X, Y],$$

since ∇ has no torsion and the Lie bracket is $[X, Y] = \nabla_X Y - \nabla_Y X$. On the other hand, one can find that A_F^{-1} is just the shape operator of the Wulff shape W_F in \mathbb{R}^{n+1} and thus by the Codazzi equation, we have

$$(2.8) \quad \nabla_X(A_F^{-1}Y) - A_F^{-1}(\nabla_X Y) = \nabla_Y(A_F^{-1}X) - A_F^{-1}(\nabla_Y X),$$

where the tangent vectors of M are identified with those of \mathbb{S}^n under the Gauss map and then with those of W_F under ϕ . Then combining (2.7) and (2.8), we can get

$$(\lambda_i I - S_F)[X, Y] = 0.$$

Hence $[X, Y]$ is in D_i which proves that D_i is integrable.

Now for $\lambda_i \neq 0$, we consider the anisotropic parallel translation x_t of x with $t = 1/\lambda_i$. From (2.2) we find that dx_t vanishes on D_i and has constant rank $n - m_i$. Therefore, $x_t(M)$ is an $(n - m_i)$ -dimensional submanifold in \mathbb{R}^{n+1} and x_t maps each leaf $L_i(p)$ of D_i to one point, say $q \in x_t(M)$, *i.e.*,

$$x + t\phi \circ \nu \equiv q, \quad \text{for } x|_{L_i(p)},$$

or equivalently,

$$(2.9) \quad \lambda_i(x - q) \equiv \phi \circ \nu, \quad \text{for } x|_{L_i(p)}.$$

Notice that when limited on $L_i(p)$, ν is also a unit normal vector of $x_t(M)$ at q under parallel translations and thus defines a map

$$(2.10) \quad \nu : L_i(p) \rightarrow \mathbb{S}^{m_i} \subset \mathcal{T}_q^\perp(x_t(M)).$$

Finally, the following lemma together with formula (2.9) will complete the proof of the second part of the proposition. \square

Lemma 2.1. *The map $\nu : L_i(p) \rightarrow \mathbb{S}^{m_i}$ defined in (2.10) when $\lambda_i \neq 0$ is an open map. In particular, if M is complete, it is a diffeomorphism.*

Proof. It suffices to prove that the map $\nu : L_i(p) \rightarrow \mathbb{S}^{m_i}$ is always nondegenerate, *i.e.*,

$$(2.11) \quad d\nu(X) \neq 0, \quad \text{for any } 0 \neq X \in D_i.$$

On the other hand, we have

$$d\nu(X) = -T(X) = -A_F^{-1}S_F(X), \quad \text{for } X \in \mathcal{T}M,$$

which immediately verifies (2.11) to be correct. \square

3. PROOF OF THE MAIN RESULT

In this section, we will prove Theorem 1.1 based on a Cartan-type identity which forces the number g of distinct constant anisotropic principal curvatures must be less than 3.

Let $x : M \rightarrow \mathbb{R}^{n+1}$ be a complete anisotropic isoparametric hypersurface with g distinct constant anisotropic principal curvatures $\lambda_1, \dots, \lambda_g$ of multiplicities m_1, \dots, m_g . Then we have

Lemma 3.1. *$g \leq 2$, and one anisotropic principal curvature must be 0 if $g = 2$.*

Proof. As A_F is a positive self-dual operator, its square root C_F is uniquely determined and is also a positive self-dual operator. Without loss of generality, we can choose a local orthonormal basis e_1, \dots, e_n of $x : M \rightarrow \mathbb{R}^{n+1}$ such that under this basis,

$$C_F T C_F = \text{diag}(\lambda_1 I_{m_1}, \dots, \lambda_g I_{m_g}) =: \Lambda,$$

where T is the shape operator (matrix) of M .

Let $(\varepsilon_1, \dots, \varepsilon_n) := (C_F(e_1), \dots, C_F(e_n)) = (e_1, \dots, e_n)C_F$. Then it is a tangent frame (not necessarily orthonormal) of M , and

$$(3.1) \quad (S_F(\varepsilon_1), \dots, S_F(\varepsilon_n)) = (e_1, \dots, e_n)A_F T C_F = (\varepsilon_1, \dots, \varepsilon_n)\Lambda,$$

which shows that $\varepsilon_1, \dots, \varepsilon_n$ are the eigenvectors corresponding to the anisotropic principal curvatures, *i.e.*, they span the distributions D_1, \dots, D_g sequentially.

Now without loss of generality, suppose $g \geq 2$ and $\lambda_1 > 0$ be the smallest positive anisotropic principal curvature. Then we will show that $g = 2$ and $\lambda_2 = 0$. For $t = 1/\lambda_1$, consider the degenerate anisotropic parallel translation $x_t : M \rightarrow \mathbb{R}^{n+1}$ of x . As we showed in the proof of Proposition 2.1, $x_t(M)$ is an $(n - m_1)$ -dimensional submanifold immersed in \mathbb{R}^{n+1} whose tangent space is spanned by D_2, \dots, D_g up to translations². Thus we would like to study the geometry of $x_t(M)$ by its second fundamental form. It follows from (2.2) and (3.1) that

$$(3.2) \quad (\langle dx_t(\varepsilon_a), dx_t(\varepsilon_b) \rangle) = (I - t\tilde{\Lambda})\widetilde{A_F}(I - t\tilde{\Lambda}),$$

where the indices $a, b, \dots \in \{m_1 + 1, \dots, n\}$, $\tilde{\Lambda} := \text{diag}(\lambda_2 I_{m_2}, \dots, \lambda_g I_{m_g})$, $\widetilde{A_F} := (\langle \varepsilon_a, \varepsilon_b \rangle) = (A_F)_{ab}$ is the submatrix of A_F with respect to $\text{Span}(e_{m_1+1}, \dots, e_n)$ and thus is positive. Let $\widetilde{C_F}$ be the square root of $\widetilde{A_F}$. Then it can be easily verified from (3.2) that

$$(\tilde{e}_{m_1+1}, \dots, \tilde{e}_n) := (\varepsilon_{m_1+1}, \dots, \varepsilon_n)(I - t\tilde{\Lambda})^{-1}\widetilde{C_F}^{-1}$$

²In general, D_1 is not the normal space of $x_t(M)$.

is an orthonormal basis of $x_t(M)$ under the induced metric. From now on, we would like to use upper indices to denote elements of the inverse of a matrix. For example, we write $\widetilde{C}_F^{-1} = (\widetilde{C}_F^{ab})$. Then by (2.2), we have

$$(3.3) \quad dx_t(\tilde{e}_a) = \sum_{c=m_1+1}^n dx_t(\varepsilon_c)(1-t\lambda_c)^{-1}\widetilde{C}_F^{ca} = \sum_{c=m_1+1}^n \varepsilon_c \widetilde{C}_F^{ca}.$$

Recall that ν is a unit normal vector field of $x_t(M)$ and we have

$$(3.4) \quad -d\nu(\tilde{e}_b) = A_F^{-1}S_F(\tilde{e}_b) = \sum_{c=m_1+1}^n A_F^{-1}(\varepsilon_c)\lambda_c(1-t\lambda_c)^{-1}\widetilde{C}_F^{cb}.$$

Taking inner product of (3.3) and (3.4), we obtain the second fundamental form II_ν of $x_t(M)$ in direction ν as follows:

$$II_\nu(\tilde{e}_a, \tilde{e}_b) = \langle dx_t(\tilde{e}_a), -d\nu(\tilde{e}_b) \rangle = \widetilde{C}_F^{ca} \lambda_c (1-t\lambda_c)^{-1} \widetilde{C}_F^{cb},$$

or in matrix form,

$$(3.5) \quad II_\nu = \widetilde{C}_F^{-1} \frac{\widetilde{\Lambda}}{I - t\widetilde{\Lambda}} \widetilde{C}_F^{-1}.$$

For any point $q \in x_t(M)$ and any unit normal vector $u \in \mathbb{S}^{m_1} \subset \mathcal{V}_q(x_t(M))$, by Proposition 2.1 and Lemma 2.1, there exists a unique point p in the leaf $L_1 = x_t^{-1}(q) = W_F^{m_1}$ of the distribution D_1 such that $\nu(p) = u$. Hence, for $u, -u \in \mathbb{S}^{m_1} \subset \mathcal{V}_q(x_t(M))$, there exist $p_1, p_2 \in L_1$ such that $\nu(p_1) = u$ and $\nu(p_2) = -u$. Therefore, both II_u and II_{-u} can be expressed in the form (3.5), although the matrix \widetilde{C}_F^{-1} may differ from each other and thus we denote them simply by $\widetilde{C}_{F1}^{-1}, \widetilde{C}_{F2}^{-1}$. On the other hand, we have $tr(II_{-u}) = -tr(II_u)$. Thus we derive the following Cartan-type identity:

$$\sum_{a=m_1+1}^n (\widetilde{A}_{F1}^{aa} + \widetilde{A}_{F2}^{aa}) \frac{\lambda_a}{1-t\lambda_a} = 0,$$

or equivalently,

$$(3.6) \quad \sum_{k=2}^g \frac{\Gamma_F^k \lambda_k}{1-t\lambda_k} = 0,$$

where $\widetilde{A}_{Fl}^{aa} = (\widetilde{C}_{Fl}^{-2})^{aa}$ s are the diagonal elements of \widetilde{A}_F^{-1} corresponding to the point p_l , $l = 1, 2$, $\Gamma_F^k := \sum_{a=n_{k-1}+1}^{n_k} (\widetilde{A}_{F1}^{aa} + \widetilde{A}_{F2}^{aa}) > 0$, and $n_k = \sum_{j=1}^k m_j$, $k = 1, \dots, g$.

Recall that λ_1 is the smallest positive anisotropic principal curvature and $t = 1/\lambda_1$. Consequently each term in the summation of (3.6) is non-positive and thus must vanish, which implies that $g = 2$ and $\lambda_2 = 0$. The proof is now completed. \square

An immediate consequence of (3.5) and Lemma 3.1 is the following.

Corollary 3.1. *When $g = 2$, $x_t(M)$ ($t = 1/\lambda_1$) is totally geodesic and thus it is congruent to $\mathbb{R}^{n-m_1} \subset \mathbb{R}^{n+1}$.*

Remark 3.1. As the way taken in [5], one can also calculate the power expansion of F -Weingarten operator S_F^t defined in (2.3) with respect to t for t sufficiently close to $1/\lambda_1$, so as to deduce the Cartan-type identity (3.6) and hence Lemma 3.1 and Corollary 3.1.

Finally, combining Lemma 3.1, Proposition 1.1, Corollary 3.1 and formulas (2.9), (1.4), we can conclude the classification in Theorem 1.1. In view of results in Section 2, it seems that the local version of this classification should also hold for analytic anisotropic defining functions since in this case one may possibly derive the Cartan-type identity (3.6) by analytic extension.

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